Table of Contents

Trying to solve f(f(x)) = x**2 - 2

Problem: Find $f(x)$ satisfying the functional equation

$$f(f(x)) = x^2 - 2 \quad . \qquad (1)$$

Solution:

- First, it is noteworthy that the equation
  $$f(f(x)) = x^2 \qquad (2)$$
  has simple solutions
  $$f_0(x) = x^{\pm\sqrt{2}} \quad . \qquad (3)$$
- One can try to reduce (1) to (2) by changing "coordinates" $x \to y$ with
  $$y = \varphi(x) \quad . \qquad (4)$$
- In y coordinates, eq.(1) can be rewritten as
  $$f_y(f_y(y)) = \varphi(\varphi_{-1}^2(y) - 2) \quad , \qquad (5)$$
  where
  $$f_y = \varphi f \varphi_{-1} \quad . \qquad (6)$$
- Requesting eq.(2) to hold for f_y gives
  $$f_y(f_y(y)) = \varphi(\varphi_{-1}^2(y) - 2) = y^2 \quad , \qquad (7)$$
  or
  $$\varphi_{-1}^2(y) - 2 = \varphi_{-1}(y^2) \quad , \qquad (8)$$
  or
  $$g^2(y) - 2 = g(y^2) \quad , \qquad (9)$$
  where we denoted
  $$g(y) = \varphi_{-1}(y) \quad , \qquad (10)$$
  Now a solution g(y) of the new functional equation (9) will transform a solution (3) of eq.(2) (more precisely, the solution $f_0(y) = y^{\pm\sqrt{2}}$ of the eq.(7)) into a solution of eq.(1):
  $$f = f_x = \varphi_{-1} f_y \varphi = g f_y g_{-1} = g f_0 g_{-1} \quad . \qquad (11)$$
- ***Looking for solutions of eq.(9)***
  - ♦ ***Assumption 1***: $g(y)$ is defined in the vicinity of $y = 0$
    - ◊ ***Assumption 2A***: $g(y)$ and its derivatives of all orders are finite at $y = 0$. Then, by comparing Taylor-series expansions for the left- and right-hand sides of eq.9, one can conclude that all the derivatives are zeros, i.e. $g(y) = const$. The corresponding solutions, $g(y) = -1$ and $g(y) = 2$, are of no interest for us, as these functions are not invertible.
    - ◊ ⇒ A singularity of $g(y)$ at $y = 0$ is needed.
      - · ***Assumption 2B***: $g(0) = \infty$
        - • ***Assumption 3***: $g(y) \approx y^{-\alpha}$ at $y \to 0$ with $\alpha > 0$
          - ♦ ***A good guess***
            $$g(y) = (1 + y^{2\alpha})/y^{\alpha} = y^{\alpha} + y^{-\alpha} \qquad (12)$$
            provides ***a solution for any*** $\alpha \neq 0$
            - ◊ Properties:
              - · defined at $(0, \infty)$
              - · has maxima $= \infty$ at $y \to 0$ and $y \to \infty$ and one minimum $g(1) = 2$
                - • ⇒ there are ***two inverse functions***, both defined in the interval $[2, \infty)$, and taking values in the intervals $(0, 1]$ and $[1, \infty)$ respectively
            - ◊ Independently of the α value, all the solutions will give the same result when fed into eq.(11), as we

will see a bit later.
Therefore, we can consider the details for just one case of e.g. $\alpha = 1$:

. $g(y) \equiv x = y + 1/y$ (12.1)

· The two inverse functions are

$g_{-1}(x) \equiv y = (x \pm \sqrt{x^2 - 4})/2$ (13.1)

or $y = [(x \pm \sqrt{x^2 - 4})/2]^{1/\alpha}$ for any α (13)

- **Feeding the found solution $g(y)$ into the formula (11)**
  According to (11) we get $f(x)$ -- the solution of eq.(1) -- in three steps:
  1. taking $g_{-1}(x)$: with eq.(13.1), the result is
     $g_{-1}(x) \equiv y = (x \pm \sqrt{x^2 - 4})/2$
  2. applying one of two functions $f_0(y) = y^{\pm\sqrt{2}}$ to the result of step 1: taking e.g. $f_0(y) = y^{\sqrt{2}}$, gives
     $z(x) = [(x \pm \sqrt{x^2 - 4})/2]^{\sqrt{2}}$
  3. finally, applying function (12.1) for the argument resulting from the step 2:
     $f(x) = z + 1/z$

- **Remarks**
  - Due to specific properties of the function $f(x) = z + 1/z$ applied on step 3, the result does not depend on which one out of the two functions $f_0(y) = y^{\pm\sqrt{2}}$ is chosen on step 2
  - If the general solution (13) is taken instead of (13,1) on step 1, then an additional exponential power $1/\alpha$ appears on step 1 and gets propagated on step 2. Then, on step 3, the general formula (13) is applied instead of (13.1), and this involves the extra α power, which cancels the $1/\alpha$ power. So the result does not change.
  - The results are identical for the plus and minus signs in the formula used on step 1. Indeed, $(x + \sqrt{x^2 - 4})/2 = [(x - \sqrt{x^2 - 4})/2]^{-1}$, Therefore, changing plus to minus just exchanges the terms z and $1/z$ on step 3.

This topic: Main > AVFedotovHowToSolveFFequalsToXsquaredMinus2
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