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Trying to solve $f(f(x)) = x^{**2} - 2$

Problem: Find $f(x)$ satisfying the functional equation

$$f(f(x)) = x^2 - 2 \quad . \quad (1)$$

Solution:

- First, it is noteworthy that the equation

$$f(f(x)) = x^2 \quad (2)$$

has simple solutions

$$f_0(x) = x^{\pm\sqrt{2}} \quad . \quad (3)$$

- One can try to reduce (1) to (2) by changing "coordinates" $x \rightarrow y$ with

$$y = \varphi(x) \quad . \quad (4)$$

- In y coordinates, eq.(1) can be rewritten as

$$f_y(f_y(y)) = \varphi(\varphi_{-1}^2(y) - 2) \quad , \quad (5)$$

where

$$f_y = \varphi f \varphi_{-1} \quad . \quad (6)$$

- Requesting eq.(2) to hold for f_y gives

$$f_y(f_y(y)) = \varphi(\varphi_{-1}^2(y) - 2) = y^2 \quad , \quad (7)$$

or

$$\varphi_{-1}^2(y) - 2 = \varphi_{-1}(y^2) \quad , \quad (8)$$

or

$$g^2(y) - 2 = g(y^2) \quad , \quad (9)$$

where we denoted

$$g(y) = \varphi_{-1}(y) \quad , \quad (10)$$

Now a solution $g(y)$ of the new functional equation (9) will transform a solution (3) of eq.(2) (more precisely, the solution $f_0(y) = y^{\pm\sqrt{2}}$ of the eq.(7)) into a solution of eq.(1):

$$f = f_x = \varphi_{-1} f_y \varphi = g f_y g_{-1} = g f_0 g_{-1} \quad . \quad (11)$$

- **Looking for solutions of eq.(9)**

- ◆ **Assumption 1:** $g(y)$ is defined in the vicinity of $y = 0$

- ◇ **Assumption 2A:** $g(y)$ and its derivatives of all orders are finite at $y = 0$. Then, by comparing Taylor-series expansions for the left- and right-hand sides of eq.9, one can conclude that all the derivatives are zeros, i.e. $g(y) = \text{const}$. The corresponding solutions, $g(y) = -1$ and $g(y) = 2$, are of no interest for us, as these functions are not invertible.

- ◇ \Rightarrow A singularity of $g(y)$ at $y = 0$ is needed.

- **Assumption 2B:** $g(0) = \infty$

- **Assumption 3:** $g(y) \approx y^{-\alpha}$ at $y \rightarrow 0$ with $\alpha > 0$

- ◆ **A good guess**

$$g(y) = (1 + y^{2\alpha})/y^\alpha = y^\alpha + y^{-\alpha} \quad (12)$$

provides **a solution for any** $\alpha \neq 0$

◇ Properties:

- defined at $(0, \infty)$

- has maxima $= \infty$ at $y \rightarrow 0$ and $y \rightarrow \infty$

- and one minimum $g(1) = 2$

- \Rightarrow there are **two inverse functions**, both defined in the interval $[2, \infty)$, and taking values in the intervals $(0, 1]$ and $[1, \infty)$ respectively

◇ Independently of the α value, all the solutions will give the same result when fed into eq.(11), as we

will see a bit later.

Therefore, we can consider the details for just one case of e.g. $\alpha = 1$:

$$g(y) \equiv x = y + 1/y \quad (12.1)$$

The two inverse functions are

$$g_{-1}(x) \equiv y = (x \pm \sqrt{x^2 - 4})/2 \quad (13.1)$$

$$\text{or } y = [(x \pm \sqrt{x^2 - 4})/2]^{1/\alpha} \text{ for any } \alpha \quad (13)$$

• **Feeding the found solution $g(y)$ into the formula (11)**

According to (11) we get $f(x)$ -- the solution of eq.(1) -- in three steps:

1. taking $g_{-1}(x)$: with eq.(13.1), the result is

$$g_{-1}(x) \equiv y = (x \pm \sqrt{x^2 - 4})/2$$
2. applying one of two functions $f_0(y) = y^{\pm\sqrt{2}}$ to the result of step 1: taking e.g. $f_0(y) = y^{\sqrt{2}}$, gives

$$z(x) = [(x \pm \sqrt{x^2 - 4})/2]^{\sqrt{2}}$$
3. finally, applying function (12.1) for the argument resulting from the step 2:

$$f(x) = z + 1/z$$

• **Remarks**

- ◆ Due to specific properties of the function $f(x) = z + 1/z$ applied on step 3, the result does not depend on which one out of the two functions $f_0(y) = y^{\pm\sqrt{2}}$ is chosen on step 2
- ◆ If the general solution (13) is taken instead of (13.1) on step 1, then an additional exponential power $1/\alpha$ appears on step 1 and gets propagated on step 2. Then, on step 3, the general formula (13) is applied instead of (13.1), and this involves the extra α power, which cancels the $1/\alpha$ power. So the result does not change.
- ◆ The results are identical for the plus and minus signs in the formula used on step 1. Indeed, $(x + \sqrt{x^2 - 4})/2 = [(x - \sqrt{x^2 - 4})/2]^{-1}$, Therefore, changing plus to minus just exchanges the terms z and $1/z$ on step 3.

This topic: Main > AVFedotovHowToSolveFFequalsToXsquaredMinus2

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